Elementary Derivation of the Dirac Equation. VII

Hans Sallhofer

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The momentum in Dirac-like electrodynamics

For electrodynamics [1], (5), according to [2], (1) and [2], (5), the following energy balances hold

$$\dot{U} + \operatorname{div} \boldsymbol{S} = 0\,, \tag{1}$$

and

$$\dot{\check{U}} + \operatorname{div} \check{\boldsymbol{S}} = 0\,. \tag{2}$$

Whereas (1) reproduces the well-known equations of continuity for the two photon fields described by [1], Eq. (2) represents a completely new relation. From its derivation, as indicated between [2], (4) and [2], (5), it emerges as a peculiarity of Dirac-like electrodynamics [1], (5) or [1], (8), respectively.

Explicit evaluation of (2) yields

$$\check{U} = \varepsilon\,(E_1^{\mathrm{Im}} E_2^{\mathrm{Re}} - E_2^{\mathrm{Im}} E_1^{\mathrm{Re}}) + \mu\,(H_1^{\mathrm{Im}} H_2^{\mathrm{Re}} - H_2^{\mathrm{Im}} H_1^{\mathrm{Re}})\,, \tag{3}$$

and

$$\frac{1}{c}\,\check{\boldsymbol{S}} = \boldsymbol{H}^{\mathrm{Re}} E_3^{\mathrm{Im}} - \boldsymbol{H}^{\mathrm{Im}} E_3^{\mathrm{Re}} + \boldsymbol{E}^{\mathrm{Im}} H_3^{\mathrm{Re}} - \boldsymbol{E}^{\mathrm{Re}} H_3^{\mathrm{Im}} - \boldsymbol{e}_3 [(\boldsymbol{E}^{\mathrm{Im}} \cdot \boldsymbol{H}^{\mathrm{Re}}) - (\boldsymbol{E}^{\mathrm{Re}} \cdot \boldsymbol{H}^{\mathrm{Im}})]\,. \tag{4}$$

By introducing a vector

$$\boldsymbol{V} = \varepsilon\,(\boldsymbol{E}^{\mathrm{Im}} \times \boldsymbol{E}^{\mathrm{Re}}) + \mu\,(\boldsymbol{H}^{\mathrm{Im}} \times \boldsymbol{H}^{\mathrm{Re}}) \quad \text{with} \quad (\boldsymbol{V} \cdot \boldsymbol{e}_3) = \check{U}\,, \tag{5}$$

and a tensor

$$\frac{1}{c}\,\mathbf{T} = \boldsymbol{H}^{\mathrm{Re}} \boldsymbol{E}^{\mathrm{Im}} - \boldsymbol{H}^{\mathrm{Im}} \boldsymbol{E}^{\mathrm{Re}} + \boldsymbol{E}^{\mathrm{Im}} \boldsymbol{H}^{\mathrm{Re}} - \boldsymbol{E}^{\mathrm{Re}} \boldsymbol{H}^{\mathrm{Im}} - \mathbf{1}[(\boldsymbol{E}^{\mathrm{Im}} \cdot \boldsymbol{H}^{\mathrm{Re}}) - (\boldsymbol{E}^{\mathrm{Re}} \cdot \boldsymbol{H}^{\mathrm{Im}})] \quad \text{with} \quad (\mathbf{T} \cdot \boldsymbol{e}_3) = \check{\boldsymbol{S}}\,, \tag{6}$$

we can write (2) as

$$(\dot{\boldsymbol{V}} + \operatorname{div} \mathbf{T}) \cdot \boldsymbol{e}_3 = 0\,. \tag{7}$$

In contrast to the energy balance (1), where the time derivative involves a scalar quantity, i.e. the electromagnetic energy density, in (7), or (2) respectively, the time derivative of a vector appears. This vector is constructed from the vector products of the electric and magnetic parts of the two photons and has the direction of the common axis of revolution of both photon wave fields. Most obvious assumption: (7) represents a momentum balance, (5) denotes the angular and spin momentum of the two-photon system, and (6) the corresponding flow tensor. The balance (7) gives an immediate basis to the model concepts below [3], (12). Equation (7) evidently is valid for an electromagnetic momentum vortex with a special direction: its axis points into the 3-axis. – The balance will be now established in a completely general way with the help of Dirac-like electrodynamics [1], (5).

First one computes the divergence of the flow tensor from (6):

$$\begin{aligned}\frac{1}{c} \operatorname{div} \mathbf{T} = {} & (\operatorname{div} \boldsymbol{H}^{\mathrm{Re}})\, \boldsymbol{E}^{\mathrm{Im}} + (\boldsymbol{H}^{\mathrm{Re}} \cdot \operatorname{grad})\, \boldsymbol{E}^{\mathrm{Im}} \\ & - (\operatorname{div} \boldsymbol{H}^{\mathrm{Im}})\, \boldsymbol{E}^{\mathrm{Re}} - (\boldsymbol{H}^{\mathrm{Im}} \cdot \operatorname{grad})\, \boldsymbol{E}^{\mathrm{Re}} \\ & + (\operatorname{div} \boldsymbol{E}^{\mathrm{Im}})\, \boldsymbol{H}^{\mathrm{Re}} + (\boldsymbol{E}^{\mathrm{Im}} \cdot \operatorname{grad})\, \boldsymbol{H}^{\mathrm{Re}} \\ & - (\operatorname{div} \boldsymbol{E}^{\mathrm{Re}})\, \boldsymbol{H}^{\mathrm{Im}} - (\boldsymbol{E}^{\mathrm{Re}} \cdot \operatorname{grad})\, \boldsymbol{H}^{\mathrm{Im}} \\ & - \operatorname{grad} [(\boldsymbol{E}^{\mathrm{Im}} \cdot \boldsymbol{H}^{\mathrm{Re}}) - (\boldsymbol{E}^{\mathrm{Re}} \cdot \boldsymbol{H}^{\mathrm{Im}})]\,. \end{aligned} \tag{8}$$

Because of the second line of [1], (5) the four terms containing divergences drop out. With help of the vector relation

$$\operatorname{grad}(\boldsymbol{a} \cdot \boldsymbol{b}) = (\boldsymbol{b} \cdot \operatorname{grad})\, \boldsymbol{a} + (\boldsymbol{a} \cdot \operatorname{grad})\, \boldsymbol{b} + (\boldsymbol{a} \times \operatorname{rot} \boldsymbol{b}) + (\boldsymbol{b} \times \operatorname{rot} \boldsymbol{a}) \tag{9}$$

one can see that four more terms in (8) cancel with the last one. Thus the following relation remains:

$$\frac{1}{c} \operatorname{div} \mathbf{T} + (\boldsymbol{E}^{\mathrm{Im}} \times \operatorname{rot} \boldsymbol{H}^{\mathrm{Re}}) + (\boldsymbol{H}^{\mathrm{Re}} \times \operatorname{rot} \boldsymbol{E}^{\mathrm{Im}}) - (\boldsymbol{E}^{\mathrm{Re}} \times \operatorname{rot} \boldsymbol{H}^{\mathrm{Im}}) - (\boldsymbol{H}^{\mathrm{Im}} \times \operatorname{rot} \boldsymbol{E}^{\mathrm{Re}}) = 0\,. \tag{10}$$

Here the last four terms immediately yield the momentum rate (5) because of the first line of [1], (5), so that the momentum balance (7) follows in complete generality as

$$\dot{\boldsymbol{V}} + \operatorname{div} \mathbf{T} = 0\,. \tag{11}$$

The direct derivation of the momentum balance (11) from the two photon electrodynamics once again shows the Dirac structure to be a quite central system-immanence of electrodynamics.

Reprint requests to Prof. H. Sallhofer, Fischerstr. 12, A-5280 Braunau, Österreich.

[1] H. Sallhofer, Z. Naturforsch. **33a**, 1378 (1978).
[2] H. Sallhofer, Z. Naturforsch. **34a**, 1145 (1979).
[3] H. Sallhofer, Z. Naturforsch. **35a**, 995 (1980).

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